NON-ORTHOGONAL BASES

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Wednesday, September 29, 2021
Assignment 4 Assigned later this week
- Will include a programming components
- Due October 11, 2021
WHAT’S ON THE AGENDA FOR TODAY?

Last time: Non-Orthobases
  - Isomorphism of separable Hilbert space $\mathcal{H}$ to $\ell_2$
  - Non-orthobases may compromise stability!

Today: More on non-orthobases
  - Dual basis
  - Infinite dimension

Monday October 04, 2021: Least-square regression

Reading: Romberg, lecture notes 7
Proposition. Let \( \{v_i\}_{i=1}^n \) be a linearly independent set in a Hilbert space \( \mathcal{H} \) of dimension \( n \). Then, for any \( x \in \mathcal{H}, x = \sum_{i=1}^n \alpha_i v_i \) for some \( \alpha \in \mathbb{R}^n \). In addition, there exists \( A, B > 0 \) such that

\[
A \| \alpha \|_2^2 \leq \| x \|_{\mathcal{H}}^2 \leq B \| \alpha \|_2^2
\]

Inequality is tight on both sides.

For orthobases, \( A = B = 1 \)

Interpretation:
- The values of \( A \) and \( B \) govern the stability of the representation.

Examples
Recall from orthobases:
- perfectly stable representation $A = B = 1$
- Efficient computation of representations: $\alpha_i = \langle x, v_i \rangle$

**Proposition.** For any $x \in \mathcal{H}$ with basis $\{v_i\}_{i=1}^n$ we have

$$x = \sum_{i=1}^n \alpha_i v_i \quad \text{with} \quad \alpha = G^{-1} \begin{bmatrix} \langle x, v_1 \rangle \\ \langle x, v_2 \rangle \\ \vdots \\ \langle x, v_n \rangle \end{bmatrix}$$

There also exists a basis $\{\tilde{v}_i\}_{i=1}^n$ such that $\alpha_i = \langle x, \tilde{v}_i \rangle$
Definition. \( \{v_i\}_{i=1}^{\infty} \) is a Riesz basis for Hilbert space \( \mathcal{H} \) if \( \text{cl}(\text{span}(\{v_i\}_{i=1}^{\infty})) = \mathcal{H} \) and there exists \( A, B > 0 \) such that

\[
A \sum_{i=1}^{\infty} \alpha_i^2 \leq \left\| \sum_{i=1}^{n} \alpha_i v_i \right\|_{\mathcal{H}}^2 \leq B \sum_{i=1}^{\infty} \alpha_i^2
\]

uniformly for all sequences \( \{\alpha_i\}_{i\geq 1} \) with \( \sum_{i\geq 1} \alpha_i^2 < \infty \).

In infinite dimension, the existence of \( A, B > 0 \) is not automatic.

Examples
Computing expansion on Riesz basis not as simple in infinite dimension: Gram matrix is “infinite”

The Grammian is a linear operator

\[
G : \ell_2(\mathbb{Z}) \rightarrow \ell_2(\mathbb{Z}) : x \mapsto y \text{ with } [G(x)]_n \triangleq y_n = \sum_{\ell=-\infty}^{\infty} \langle v_\ell, v_n \rangle x_\ell
\]

Fact: there exists another linear operator \( H : \ell_2(\mathbb{Z}) \rightarrow \ell_2(\mathbb{Z}) \) such that

\[
H(G(x)) = x
\]

We can replicate what we did in finite dimension!