NON-ORTHOGONAL BASES

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LOGISTICS

Assignment 4 Assigned later this week

- Will include a programming components
- Due **October 11, 2021**

WHAT'S ON THE AGENDA FOR TODAY?

Last time: Non-Orthobases

- Isomorphism of separable Hilbert space $\mathcal{H}$ to ℓ_2
- Non-orthobases may compromise stability!

Today: More on non-orthobases

- Dual basis
- Infinite dimension

Monday October 04, 2021: Least-square regression

Reading: Romberg, lecture notes 7

RECAP: NON-ORTHOGONAL BASES IN FINITE DIMENSION

> **Proposition.** Let $\{v_i\}_{i=1}^n$ be a linearly independent set in a Hilbert space $\mathcal{H}$ of dimension n. Then, for any $x \in \mathcal{H}$, $x = \sum_{i=1}^n \alpha_i v_i$ for some $\boldsymbol{\alpha} \in \mathbb{R}^n$. In addition, there exists $A, B > 0$ such that
>
> $$A \|\boldsymbol{\alpha}\|_2^2 \leq \|\boldsymbol{x}\|_{\mathcal{H}}^2 \leq B \|\boldsymbol{\alpha}\|_2^2$$

Inequality is tight on both sides

For orthobases, $A = B = 1$

Interpretation:

- The values of A and B govern the *stability* of the representation

Examples

NON-ORTHOGONAL BASES IN FINITE DIMENSION: DUAL BASIS

Recall from orthobases:

- perfectly stable representation $A = B = 1$
- Efficient computation of representations: $\alpha_i = \langle x, v_i \rangle$

Proposition. For any $x \in \mathcal{H}$ with basis $\{v_i\}_{i=1}^n$ we have

$$x = \sum_{i=1}^{n} \alpha_i v_i \qquad \text{with} \qquad \boldsymbol{\alpha} = \mathbf{G}^{-1} \begin{bmatrix} \langle x, v_1 \rangle \\ \langle x, v_2 \rangle \\ \vdots \\ \langle x, v_n \rangle \end{bmatrix}$$

There also exists a basis $\{\tilde{v}_i\}_{i=1^n}$ such that $\alpha_i = \langle x, \tilde{v}_i \rangle$

NON-ORTHOGONAL BASES IN INFINITE DIMENSION

Definition.
$\{v_i\}_{i=1}^{\infty}$ is a **Riesz basis** for Hilbert space $\mathcal{H}$ if $\mathrm{cl}(\mathrm{span}(\{v_i\}_{i=1}^{\infty})) = \mathcal{H}$ and there exists $A, B > 0$ such that

$$A \sum_{i=1}^{\infty} \alpha_i^2 \leq \left\| \sum_{i=1}^{n} \alpha_i v_i \right\|_{\mathcal{H}}^2 \leq B \sum_{i=1}^{\infty} \alpha_i^2$$

uniformly for all sequences $\{\alpha_i\}_{i\geq 1}$ with $\sum_{i\geq 1} \alpha_i^2 < \infty$.

In infinite dimension, the existence of $A, B > 0$ is **not** automatic.

Examples

NON-ORTHOGONAL BASES IN FINITE DIMENSION: DUAL BASIS

Computing expansion on Riesz basis not as simple in infinite dimension: Gram matrix is “infinite”

The Grammiam is a **linear operator**

$$\mathcal{G} : \ell_2(\mathbb{Z}) \to \ell_2(\mathbb{Z}) : \mathbf{x} \mapsto \mathbf{y} \text{ with } [\mathcal{G}(\mathbf{x})]_n \triangleq y_n = \sum_{\ell=-\infty^\infty} \langle v_\ell, v_n \rangle x_\ell$$

Fact: there exists another linear operator $\mathcal{H} : \ell_2(\mathbb{Z}) \to \ell_2(\mathbb{Z})$ such that

$$\mathcal{H}(\mathcal{G}(\mathbf{x})) = \mathbf{x}$$

We can replicate what we did in finite dimension!