REPRODUCING KERNEL HILBERT SPACES

Dr. Matthieu R Bloch

Wednesday October 20, 2021

LOGISTICS

Drop date: October 30, 2021

More office hours

- Tuesdays 8am-9am on BlueJeans (https://bluejeans.com/205357142)
- Come prepared!

Midterm 2: initially scheduled for Wednesday November 3, 2021

- Moved to Monday November 8, 2021 (gives you weekend to prepare)
- Coverage: everything since Midterm 1 (dont' forget the fundamentals though), emphasis on **regression**

WHAT'S ON THE AGENDA FOR TODAY?

Last time:

- Motivation for RKHS
- Functional on Hilbert spaces

Today:

- Reproducing Kernel Hilbert Spaces

Reading: Romberg, lecture notes 10

LINEAR FUNCTIONALS ON HILBERT SPACES

In what follows, $\mathcal{F}$ is a Hilbert space with scalar field $\mathbb{R}$

> **Definition.**
>
> A functional $F : \mathcal{F} \to \mathbb{R}$ associates real-valued number to an element of a Hilbert space $\mathcal{F}$

Notation can be tricky when the Hilbert space is a space of functions: F can act on a function $f \in \mathcal{F}$

Examples

$$F : \mathcal{F} \to \mathbb{R} : x \mapsto \|x\|$$

$$F_c : \mathcal{F} \to \mathbb{R} : x \longmapsto \langle x, c \rangle_{\mathcal{F}} \quad \text{for some } c \in \mathcal{F}$$

$$F : L_2(\mathbb{R}) \to \mathbb{R} : f \longmapsto \underbrace{\int_{-\infty}^{+\infty} f(t)\,\omega(t)\,dt}_{\langle f, \omega \rangle} \quad \text{for some } \omega(t) \in \mathbb{R}$$

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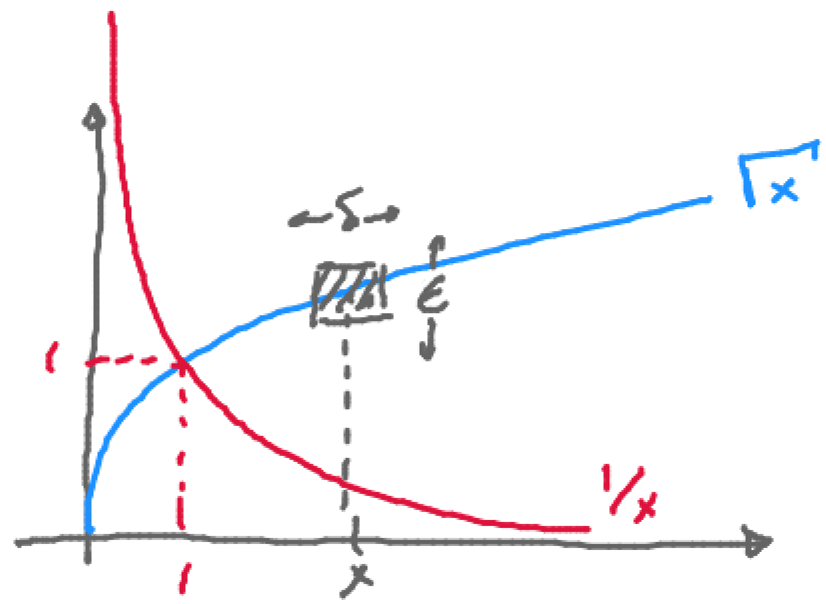
Examples

> **Definition.**
>
> A functional $F : \mathcal{F} \to \mathbb{R}$ is continuous at $x \in \mathcal{F}$ if
>
> $$\forall \epsilon > 0 \exists \delta > 0 \text{ such that } \|x - y\|_{\mathcal{F}} \leq \delta \Rightarrow |F(x) - F(y)| \leq \epsilon \qquad \cancel{\forall x, y \in \mathcal{F}}$$
>
> depends on x ?
>
> If this is true for every $x \in \mathcal{F}$, F is continuous.

Warning: I wasn't careful enough last time in the definition of continuity uniform continuity: F is uniformly continuous (everywhere) if

$$\forall \epsilon > 0 \ \underbrace{\exists \delta > 0}_{\text{indep. of } x} \text{ s.t. } \forall x, y \in \mathcal{F} \quad \|x-y\| \leq \delta \Rightarrow |F(x) - F(y)| \leq \epsilon$$



(Wikipedia)

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>
> If this is true for every $x \in \mathcal{F}$, F is continuous.

Warning: I wasn't careful enough last time in the definition of continuity

> **Proposition.**
> 1. All norms are continuous functionals
> 2. $F : \mathcal{F} \to \mathbb{R} : x \mapsto \langle x, c \rangle$ for some $c \in \mathcal{F}$ is continuous

CONTINUOUS LINEAR FUNCTIONALS ON HILBERT SPACES

Definition.
A functional F is linear if $\forall a, b \in \mathbb{R}\ \forall x, y \in \mathcal{F}\ F(ax+by) = aF(x) + bF(y)$.

Remark. $F(0) = 0$ if F linearity

$(F(0) = F(0 \cdot x) = 0 \cdot F(x) = 0$

$\in \mathcal{F}$ $\in \mathbb{R}$ $\in \mathcal{F}$

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> A functional F is linear if $\forall a, b \in \mathbb{R} \, \forall x, y \in \mathcal{F} \; F(ax + by) = aF(x) + bF(y)$.

Continuous linear functions are much more constrained than one would imagine

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> **Definition.**
> A linear functional $F : \mathcal{F} \to \mathbb{R}$ is bounded if there exists $M > 0$ such that
>
> $$\forall x \in \mathcal{F} \quad |F(x)| \leq M \,||\, x \,||\,_{\mathcal{F}}$$

> **Proposition.** A linear functional on a Hilbert space that is co~~u~~ntinuous at 0 is bounded.

Proof: Since F is continuous at 0, $\forall \epsilon > 0 \ \exists \delta > 0$ s.t. $\underbrace{\|0 - y\|_{\mathcal{F}} \leq \delta}_{\|y\| \leq \delta} \Rightarrow \underbrace{|F(0) - F(y)| \leq \epsilon}_{|F(y)| \leq \epsilon}$

In particular, we can choose $\epsilon = 1$; we know that $\exists \ \delta > 0$ s.t. $\|y\| \leq \delta \Rightarrow F(y) \leq 1$

Hence $\forall x \in \mathcal{F} \setminus \{0\}$ $|F(x)| = \left| F\left(x \cdot \frac{\delta}{\|x\|} \cdot \underbrace{\frac{\|x\|}{\delta}}_{\in \mathbb{R}}\right)\right| = \frac{\|x\|}{\delta} \left| F\Big(\underbrace{x \cdot \frac{\delta}{\|x\|}}_{\triangleq\, z \text{ s.t. } \|z\| = \delta \frac{\|x\|}{\|x\|} = \delta}\Big)\right| \leq \frac{\|x\|}{\delta}$

Therefore $\exists \ M > 0$ $(M = 1/\delta)$ s.t. $\forall x \in \mathcal{F}$ $|F(x)| \leq M \cdot \|x\|$

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A functional F is linear if $\forall a, b \in \mathbb{R}\ \forall x, y \in \mathcal{F}\ F(ax + by) = aF(x) + bF(y)$.

Continuous linear functions are much more constrained than one would imagine

Definition.
A linear functional $F : \mathcal{F} \to \mathbb{R}$ is bounded if there exists $M > 0$ such that

$$\forall x \in \mathcal{F} \quad |F(x)| \leq M \parallel x \parallel_{\mathcal{F}}$$

Proposition. A linear functional on a Hilbert space that is countinuous at 0 is bounded.

Definition.

For a linear functional $F : \mathcal{F} \to \mathbb{R}$, the following statements are equivalent:

1. F is continuous at 0
2. F is continuous at some point $x \in \mathcal{F}$
3. F is continuous everywhere on $\mathcal{F}$
4. F is uniformly continuous everywhere on $\mathcal{F}$

Proof: We know $(4) \Rightarrow (3) \Rightarrow (2)$ $\quad (4) \Rightarrow (1)$

Let's show that $(1) \Rightarrow (4)$ [F continuous at 0 $\Rightarrow$ F uniformly continuous]

$\forall\, x, y \in \mathcal{H}$ $\quad |F(x) - F(y)| = |F(x-y)|$ by linearity

$$= |F(\underbrace{x-y}_{\in \mathcal{H}}) - F(0)| \quad \text{by linearity } (F(0) = 0)$$

Since F is continuous at 0, $\forall \epsilon\ \exists\ \delta_0$ st $\|y\| \leq \delta_0 \Rightarrow |F(y)| \leq \epsilon$

$$|F(x) - F(y)| = |F(x-y) - F(0)| \leq \epsilon \quad \text{if} \quad \|x-y\| \leq \delta_0$$

REPRESENTATION OF (CONTINUOUS) LINEAR FUNCTIONALS

> **Proposition.**
>
> Let $F : \mathcal{F} \to \mathbb{R}$ be a linear functional on an n-dimensional Hilbert space $\mathcal{F}$.
>
> Then there exists $c \in \mathcal{F}$ such that $F(x) = \langle x, c \rangle$ for every $x \in \mathcal{F}$

Linear functional over finite dimensional Hilbert spaces are continuous!

Proof: Let $\{\psi_i\}_{i=1}^n$ be a orthobasis

For any vector $x \in \mathcal{H}$ $\quad x \hat{=} \sum_{i=1}^n \langle x, \psi_i \rangle \psi_i$

Then $F(x) = F\left(\sum_{i=1}^n \underbrace{\langle x, \psi_i \rangle}_{\in \mathbb{R}} \underbrace{\psi_i}_{\in \mathcal{H}}\right) = \sum_{i=1}^n \langle x, \psi_i \rangle \underbrace{F(\psi_i)}_{\in \mathbb{R}} = \Big\langle x, \underbrace{\sum_{i=1}^n F(\psi_i)\psi_i}_{\in \mathcal{H} \text{ only depends on } F \text{ and } \{\psi_i\}_{i=1}^n} \Big\rangle$

Hence $\exists\, c = \sum_{i=1}^n F(\psi_i)\psi_i$ st $\forall x \in \mathcal{H}$ $F(x) = \langle x, c \rangle$

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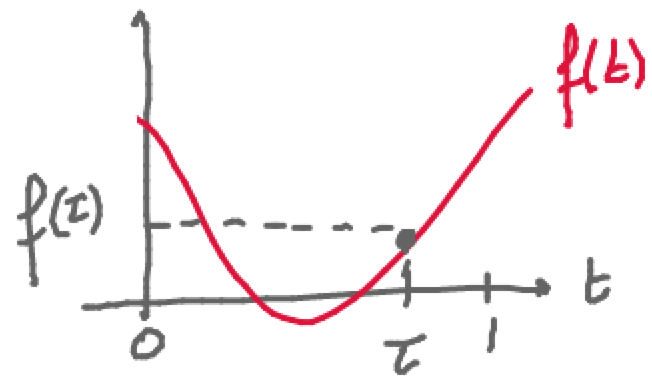
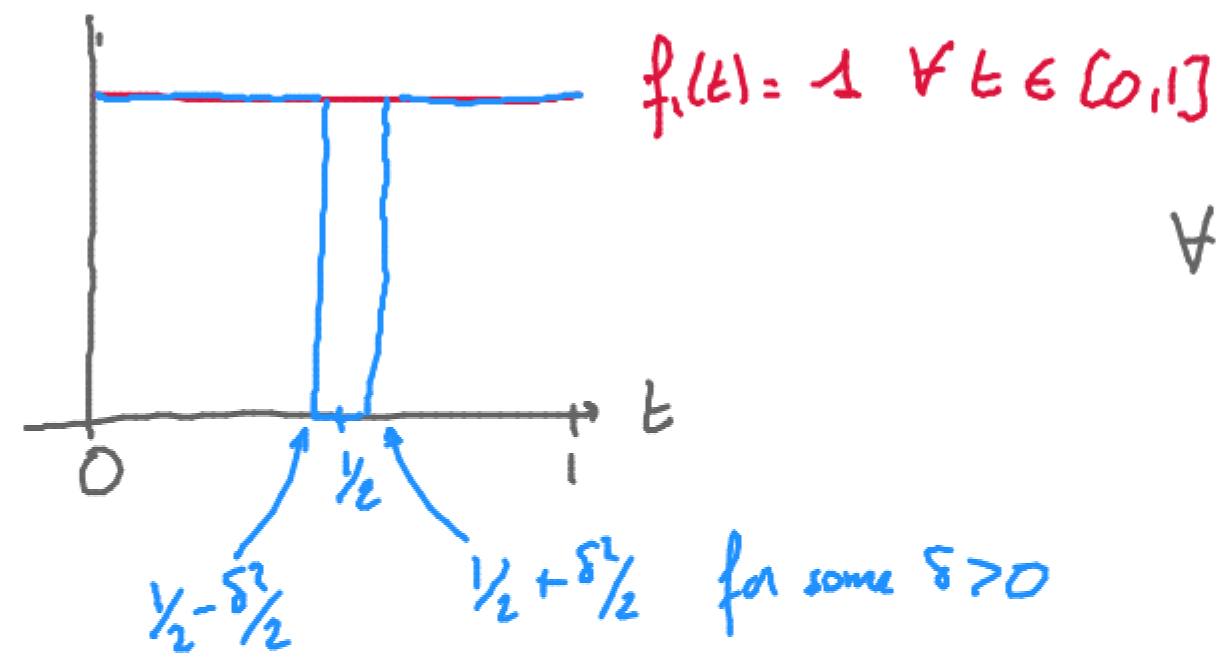
This is *not* true in infinite dimension

Example. Consider $\mathcal{F} = L_2([0;1])$ and $F_\tau \triangleq$ sampling operation at τ



$\forall \tau \in [0,1]$ $\quad F_\tau$ is linear $\quad \left(F_\tau(af+bg) = a\,f(\tau) + b\,g(\tau) \right)$

Consider the following functions.



$\forall \delta > 0 \quad F_{1/2}(f_1) = 1$

$F_{1/2}(f_2) = 0$

$$\|f_1 - f_2\| = \sqrt{\int_0^1 |f_1(t) - f_2(t)|^2 \, dt} = \delta$$

$F_{1/2}$ is not continuous

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> **Theorem (Riesz representation theorem)**
>
> Let $F : \mathcal{F} \to \mathbb{R}$ be a *continuous* linear functional on a (possible infinite dimensional) separable Hilbert space $\mathcal{F}$.
>
> Then there exists $c \in \mathcal{F}$ such that $F(x) = \langle x, c \rangle$ for every $x \in \mathcal{F}$

Proof. Let $\{\psi_i\}_{i=1}^{\infty}$ be an orthobasis (which we know exists)

Let $x \in \mathcal{H}$, we can write $x = \sum_{i=1}^{\infty} \alpha_i \psi_i$ w/ $\alpha_i = \langle x, \psi_i \rangle$

Define $x_N \triangleq \sum_{i=1}^{N} \alpha_i \psi_i$; then $F(x_N) = \sum_{i=1}^{N} \alpha_i \underbrace{F(\psi_i)}_{\triangleq \beta_i}$ (b/c sum is finite)

$$= \sum_{i=1}^{N} \alpha_i \beta_i \quad (*)$$

We know that $x_N \xrightarrow[N\to\infty]{} x$; b/c F is continuous, we know that $F(x_N) \xrightarrow[N\to+\infty]{} F(x)$

We can write $F(x) = \sum_{i=1}^{\infty} \alpha_i \beta_i$ (we want to show that this is $\langle x, c \rangle$ for some $c \in \mathcal{H}$)

We know that F is bounded $\exists M > 0$ s.t. $\forall x \in \mathcal{H}$ $|F(x)| \leq M \|x\|_{\mathcal{H}}$

Here, for $x = \sum_{i=1}^{\infty} \alpha_i \psi_i$ $\quad |F(x)| = \left| \sum_{i=1}^{\infty} \alpha_i \beta_i \right| \leq M \|x\| = M \sqrt{\sum_{i=1}^{\infty} \alpha_i^2}$

Choose a specific x s.t $\begin{cases} \forall i \in [\![1, N]\!] & \alpha_i \triangleq \langle x, \psi_i \rangle = \beta_i \\ \forall i > N & \alpha_i = 0 \end{cases}$

indep. of N

Then $\left| \sum_{i=1}^{\infty} \alpha_i \beta_i \right| = \left| \sum_{i=1}^{N} \beta_i^2 \right| \leq C \sqrt{\sum_{i=1}^{N} \beta_i^2}$; hence $\sqrt{\sum_{i=1}^{N} \beta_i^2} \leq C$

Hence $\forall N \quad \sum_{i=1}^{N} \beta_i^2 \leq C^2$ and the series converges ($\{\beta_i\}$ is square summable)

Hence $\sum_{i=1}^{\infty} \beta_i \psi_i \in \mathcal{H}$

$\in \mathcal{H}$

Finally, $F(x) = \sum_{i=1}^{\infty} \alpha_i \beta_i = \left\langle \sum_{i=1}^{\infty} \alpha_i \psi_i , \sum_{i=1}^{\infty} \beta_i \psi_i \right\rangle$

REPRESENTATION OF (CONTINUOUS) LINEAR FUNCTIONALS

Proposition.

Let $F : \mathcal{F} \to \mathbb{R}$ be a linear functional on an n-dimensional Hilbert space $\mathcal{F}$.

Then there exists $c \in \mathcal{F}$ such that $F(x) = \langle x, c \rangle$ for every $x \in \mathcal{F}$

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Theorem (Riesz representation theorem)

Let $F : \mathcal{F} \to \mathbb{R}$ be a *continuous* linear functional on a (possible infinite dimensional) separable Hilbert space $\mathcal{F}$.

Then there exists $c \in \mathcal{F}$ such that $F(x) = \langle x, c \rangle$ for every $x \in \mathcal{F}$

Proposition.

If $\{\psi_n\}_{n\geq 1}$ is an orthobasis for $\mathcal{H}$, then we can construct c above as

$$c \triangleq \sum_{n=1}^{\infty} F(\psi_n)\psi_n$$

REPRODUCING KERNEL HILBERT SPACES

Definition. (Reproducing Kernel Hilbert Spaces)

An RKHS is a Hilbert space $\mathcal{H}$ of real-valued functions $f : \mathbb{R}^d \to \mathbb{R}$ in which the sampling operation $\mathcal{S}_{\boldsymbol{\tau}} : \mathcal{H} \to \mathbb{R} : f \mapsto f(\boldsymbol{\tau})$ is continuous for every $\boldsymbol{\tau} \in \mathbb{R}^d$.

In other words, for each $\boldsymbol{\tau} \in \mathbb{R}^d$, there exists $k_{\boldsymbol{\tau}} \in \mathcal{H}$ s.t.

$$f(\boldsymbol{\tau}) = \langle f, k_{\boldsymbol{\tau}} \rangle_{\mathcal{H}} \text{ for all } f \in \mathcal{H}$$

$f(x_i)$ $\quad$ $f \in \mathcal{H}$

$\in \mathbb{R}^d$

$$\langle f, x_i \rangle_{\mathcal{H}} \equiv f(x_i)$$

$x_i \in \mathcal{H}$ $\quad$ $x_i \in \mathbb{R}^d$

$$\sum |y - \langle f, x_i \rangle|^2 + \lambda \|f\|^2$$

$$\parallel$$

$f \in \text{RKHS}$

$$\underline{\sum |y - f(x_i)|^2} + \lambda \|f\|^2$$

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Definition. (Kernel)

The kernel of an RKHS is

$$k : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R} : (\mathbf{t}, \boldsymbol{\tau}) \mapsto k_{\boldsymbol{\tau}}(\mathbf{t})$$

where $k_{\boldsymbol{\tau}}$ is the element of $\mathcal{H}$ that defines the sampling at $\boldsymbol{\tau}$.

Proposition.

A (separable) Hilbert space with orthobasis $\{\psi_n\}_{n\geq 1}$ is an RKHS iff $\forall \boldsymbol{\tau} \in \mathbb{R}^d \sum_{n=1}^{\infty} |\psi_n(\tau)|^2 < \infty$

RKHS AND NON ORTHOGONAL BASIS

If $\{\phi_n\}_{n\geq 1}$ is a Riesz basis for $\mathcal{H}$, we know that every $x \in \mathcal{H}$ can be written

$$x = \sum_{n\geq 1} \alpha_n \phi_n \text{ with } \alpha_n \triangleq \langle x, \widetilde{\phi}_n \rangle$$

where $\{\widetilde{\phi}_n\}_{n\geq 1}$ is the dual basis.

Proposition.

A (separable) Hilbert space with Riesz basis $\{\phi_n\}_{n\geq 1}$ is an RKHS with kernel

$$k(\mathbf{t}, \boldsymbol{\tau}) = \sum_{n=1}^{\infty} \phi_n(\boldsymbol{\tau}) \widetilde{\phi}_n(\mathbf{t})$$

iff $\forall \boldsymbol{\tau} \in \mathbb{R}^d \sum_{n=1}^{\infty} |\phi_n(\tau)|^2 < \infty$